

# Number of vertices in graphs with locally small chromatic number and large chromatic number

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## Abstract

We discuss the minimal number of vertices in a graph with a large chromatic number such that each ball of a fixed radius in it has a small chromatic number. It is shown that for every graph  $G$  on  $\sim ((n+rc)/(c+rc))^{r+1}$  vertices such that each ball of radius  $r$  is properly  $c$ -colorable, we have  $\chi(G) \leq n$ .

## 1 Introduction

Let  $G = (V, E)$  be a graph (with no loops or multiple edges). By  $d(u, v) = d_G(u, v)$  we denote the *distance* between the vertices  $u, v \in V$ . A subset  $V_1 \subseteq V$  is *independent* if none of the edges has both endpoints in  $V_1$ . The *chromatic number*  $\chi(G)$  of  $G$  is the minimal number of colors in a proper coloring of  $G$ , that is — the minimal number of parts in a partition of  $V$  into independent subsets.

**Definition 1.1.** *Let  $r$  be a nonnegative integer. The ball of radius  $r$  with center  $v \in V$  is the set  $U_r(v, G) = \{u \in G : d(u, v) \leq r\}$ . For  $r \geq 1$ , the  $r$ -local chromatic number  $\ell_{\chi_r}(G)$  of a graph  $G$  is the maximal chromatic number of a ball of radius  $r$  in  $G$ .*

Notice that even for  $r = 1$  our definition of the local chromatic number is quite different from that introduced by Erdős et al. in [5].

By a well-known result of Erdős [3], for every integer  $g > 2$  and every  $n > n_0(g)$ , there exists a graph on  $n^{2g+1}$  vertices of girth  $g$  and chromatic number greater than  $n$ ; thus for every  $r$  there exist a graph  $G$  with  $\ell_{\chi_r}(G) = 2$  and arbitrarily large  $\chi(G)$ . Later Erdős [4] conjectured that for every positive integer  $s$  there exists a constant  $c_s$  such that the chromatic number of each graph  $G$  having  $N$  vertices and containing no odd cycles of length less than  $c_s N^{1/s}$  does not exceed  $s + 1$ . This conjecture was proved by Kierstead, Szemerédi, and Trotter [6]. In fact, they have proved a more general result; we will formulate this result in terms of the following notion.

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**Definition 1.2.** Let  $n$ ,  $r$ , and  $c$  be positive integers. Denote by  $f_c(n, r)$  the maximal integer  $f$  with the following property: If  $G$  is a graph on  $f$  vertices and  $\ell_{\chi_r}(G) \leq c$  then  $\chi(G) \leq n$ .

Then the aforementioned result can be formulated as

$$f_c(k(c-1) + 1, r) \geq \left\lfloor \frac{r}{2k} \right\rfloor^k, \quad (1.1)$$

while the result by Erdős yields

$$f_c(n, r) \leq f_2(n, r) < n^{4r+5} \quad \text{for all } n > n_0(r). \quad (1.2)$$

Several examples (cf., for instance, [10, 11]) show that the estimate (1.1) has a correct order in  $r$  for  $c = 2$ . In [2] we show that this order is sharp even for all  $c$ . namely, it is shown that

$$f_c(k(c-1), r) < \frac{(2rc+1)^k - 1}{2r}.$$

On the other hand, the estimate (1.1) does not work for  $n \gtrsim (c-1)r$ . For  $c = 2$ , Berlov and the author [12] obtained the estimate

$$f_2(n, r) \geq \frac{(n+r+1)(n+r+2) \cdots (n+2r+1)}{2^r(r+1)^{r+1}}. \quad (1.3)$$

It is worth mentioning that for several specific series of parameters there exist almost tight bounds of  $f_c(n, r)$ . Firstly, asymptotics of  $f_2(n, 1)$  is tightly connected with the asymptotics of Ramsey numbers  $R(n, 3)$ . In the papers of Ajtai, Komlós, and Szemerédi [1] and Kim [8] it is shown that  $c_1 \frac{n^2}{\log n} \leq R(n, 3) \leq c_2 \frac{n^2}{\log n}$  for some absolute constants  $c_1, c_2$ . One can check that these results imply the bounds

$$c_3 n^2 \log n \leq f(n, 2) \leq c_4 n^2 \log n$$

for some absolute constants  $c_3, c_4$ .

The asymptotics of  $f_2(3, r)$  is also well investigated. From the generalized Mycielski construction by Stiebitz [11] it follows that  $f_2(3, r) < 2r^2 + 5r + 4$ . On the other hand, Jiang [7] showed that  $f_2(3, r) \geq (r-1)^2$ .

The aim of this paper is to extend this estimate for larger values of  $c$ . For the convenience, we use the notation  $n^{\overline{k}} = n(n+1) \cdots (n+k-1)$ . We prove the following results.

**Theorem 1.1.** For all positive integer  $n$ ,  $r$ , and  $c > 1$  we have

$$f_c(n, r) \geq \frac{(n/c + r/2)^{\overline{r+1}}}{(r+1)^{r+1}}. \quad (1.4)$$

## 2 Main result

For a graph  $G = (V, E)$  and a subset  $V_1 \subseteq V$ , we denote by  $G[V_1]$  the induced subgraph of  $G$  on the set  $V_1$ . For  $v \in V$ , we denote by  $S_r(v, G) = \{u \in V \mid d_G(u, v) = r\}$  the *sphere* with radius  $r$  and center  $v$ . In particular,  $S_0(v, G) = U_0(v, G) = \{v\}$ . Denote also by  $\partial_G^{\text{out}} V_1 = \{u \in V \setminus V_1 \mid \exists v \in V_1 : (u, v) \in E\}$  the *outer boundary* of a subset  $V_1 \subseteq V$ . In particular,  $S_r(v, G) = \partial_G^{\text{out}} U_{r-1}(v, G)$ .

Our estimate is based on the following lemma.

**Lemma 2.1.** *For every graph  $G = (V, E)$  and every positive integer  $r$ , there exists a decomposition  $V = U \sqcup N$  such that each connected component of  $U$  lies in some ball in  $G$  of radius  $r$ , and*

$$|N| \leq \frac{\sqrt[r+1]{|V|} - 1}{\sqrt[r+1]{|V|}} |V|. \quad (2.1)$$

*Proof.* Set  $v = |V|$ . We will construct inductively a sequence of partitions of  $V$  into nonintersecting parts,

$$V = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_s \sqcup N_s \sqcup V_s,$$

such that the following conditions are satisfied:

- (i) for all  $i = 1, \dots, s$  we have  $\partial_G^{\text{out}} U_i \subseteq N_s$ ; moreover,  $\partial_G^{\text{out}} V_s \subseteq N_s$ ;
- (ii) for every  $i = 1, 2, \dots, s$  the graph  $G[U_i]$  is contained in some ball in  $G$  of radius  $r$ ;
- (iii)  $(\sqrt[r+1]{v} - 1)(|U_1| + \cdots + |U_s|) \geq |N_s|$ .

For the base case  $s = 0$ , we may set  $V_0 = V$ ,  $N_0 = \emptyset$  (there are no sets  $U_i$  in this case).

For the induction step, suppose that the partition  $V = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_{s-1} \sqcup N_{s-1} \sqcup V_{s-1}$  has been constructed, and assume that the set  $V_{s-1}$  is nonempty. Consider the graph  $G_{s-1} = G[V_{s-1}]$  and choose an arbitrary vertex  $u \in V_{s-1}$ . Now consider the sets

$$U_0(u, G_{s-1}) = \{u\}, \quad U_1(u, G_{s-1}), \quad \dots, \quad U_{r+1}(u, G_{s-1}).$$

One of the ratios

$$\frac{|U_1(u, G_{s-1})|}{|U_0(u, G_{s-1})|}, \quad \frac{|U_2(u, G_{s-1})|}{|U_1(u, G_{s-1})|}, \quad \dots, \quad \frac{|U_{r+1}(u, G_{s-1})|}{|U_r(u, G_{s-1})|}$$

does not exceed  $\sqrt[r+1]{v}$ , since the product of these ratios is

$$|U_{r+1}(u, G_{s-1})| \leq v.$$

So, let us choose  $1 \leq m \leq r+1$  such that

$$\frac{|U_m(u, G_{s-1})|}{|U_{m-1}(u, G_{s-1})|} \leq \sqrt[r+1]{v}.$$

Now we set

$$U_s = U_{m-1}(u, G_{s-1}), \quad N_s = N_{s-1} \cup S_m(u, G_{s-1}), \quad V_s = V_{s-1} \setminus U_m(u, G_{s-1}).$$

Since the condition (i) was satisfied on the previous step, we have

$$(\partial_G^{\text{out}} V_s) \cup (\partial_G^{\text{out}} U_s) \subseteq \partial_G^{\text{out}} V_{s-1} \cup S_m(u, G_{s-1}) \subseteq N_s$$

so this condition also holds now. The condition (ii) is satisfied trivially. Finally, the choice of  $m$  and the condition (iii) for the previous step imply that

$$\begin{aligned} \sqrt[r+1]{v} \cdot |U_s| &= \sqrt[r+1]{v} \cdot |U_{m-1}(u, G_{s-1})| \geq |U_m(u, G_{s-1})|, \\ (\sqrt[r+1]{v} - 1)(|U_1| + \cdots + |U_{s-1}|) &\geq |N_{s-1}| \end{aligned}$$

and hence

$$(\sqrt[r+1]{v} - 1)(|U_1| + \cdots + |U_s|) \geq |N_{s-1}| + |U_m(u, G_{s-1})| - |U_{m-1}(u, G_{s-1})| = |N_s|.$$

Thus, the condition (iii) also holds on this step.

Continuing the construction in this manner, we will eventually come to the partition with  $V_s = \emptyset$  since the value of  $|V_s|$  strictly decreases. As the result, we obtain the partition  $V = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_s \sqcup N_s$  such that  $|N_s| \leq (\sqrt[r+1]{v} - 1)(|U_1| + \cdots + |U_s|)$ . So, setting  $U = U_1 \cup \cdots \cup U_s$  and  $N = N_s$  we get

$$\sqrt[r+1]{v} \cdot |N| \leq (\sqrt[r+1]{v} - 1)|U| + (\sqrt[r+1]{v} - 1)|N| = |V|(\sqrt[r+1]{v} - 1),$$

or  $|N| \leq |V| \frac{\sqrt[r+1]{v} - 1}{\sqrt[r+1]{v}}$ , as required.  $\square$

**Corollary 2.1.** *Setting  $v = f_c(n, r) + 1$ , we have*

$$v \geq \frac{\sqrt[r+1]{v}}{\sqrt[r+1]{v} - 1} (f_c(n - 2, k) + 1). \quad (2.2)$$

*Proof.* Let  $G = (V, E)$  be a graph on  $v$  vertices such that  $\ell_{\chi_r}(G) \leq c$  although  $\chi(G) > n$ . Applying Lemma 2.1 we get a decomposition  $V = U \sqcup N$  such that  $G[U]$  has a proper coloring in  $c$  colors. So,  $G[N]$  cannot be properly colored in  $n - c$  colors, hence  $|N| \geq f_c(n - c, r)$ , hence the relation (2.1) yields (2.2).  $\square$

The next proposition shows how to make an explicit estimate for  $f_c(n, r)$  from Corollary 2.1.

**Proposition 2.1.** *Suppose that for some integer  $n_0 \geq 1$  and real  $a$  the inequality*

$$f(m, k) \geq \frac{(a + m/c)^{\overline{r+1}}}{(r+1)^{r+1}} - 1 \quad (2.3)$$

*holds for  $m = n_0$ . Then the same estimate holds for all integer  $m \geq n_0$  with  $m - n_0 \equiv 0 \pmod{c}$ .*

*Proof.* We use the Induction on  $m$  with step  $c$ ; the base case holds by the conditions of the proposition.

Assume now that (2.3) holds for  $m = n - c$  but not for  $m = n$ . Denote  $v = f_c(n, r) + 1$ ; then we have

$$\sqrt[r+1]{v} < \sqrt[r+1]{\frac{(a + n/c + r)^{r+1}}{(r+1)^{r+1}}} = \frac{a + n/c + r}{r+1}$$

and hence

$$\frac{\sqrt[r+1]{v}}{\sqrt[r+1]{v} - 1} > \frac{a + n/c + r}{a + n/c - 1}.$$

By (2.2) this yields

$$v \geq \frac{a + n/c + r}{a + n/c - 1} \cdot \frac{(a + (n - c)/c)^{\overline{r+1}}}{(r+1)^{r+1}} = \frac{(a + n/c)^{\overline{r+1}}}{(r+1)^{r+1}},$$

which contradicts our assumption. Thus the induction step is proved.  $\square$

*Proof of Theorem 1.1.* In view of Proposition 2.1 it suffices to check (1.4) for all  $n \leq c$ . Trivially, we have  $f_c(n, r) \geq 1$ . On the other hand, by the AM–GM inequality we have

$$\left(\frac{n}{c} + \frac{r}{2}\right)^{\overline{r+1}} \leq \left(\frac{n}{c} + r\right)^{r+1} \leq (r+1)^{r+1},$$

thus

$$f_c(n, r) \geq 1 \geq \frac{(n/c + r/2)^{\overline{r+1}}}{(r+1)^{r+1}},$$

as required.  $\square$

**Remark.** For large values of parameters, one may use the larger value of  $a$  in Proposition 2.1, for instance, by the use of (1.1).

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